Final Exam for PHY1222 course. UCL, 21/08/2007

Time: 3 hours

Hilbert space

- Present in a concise way the most important formal features of an Hilbert space. List and prove the key theorems for hermitian operators and their relation to QM.

- An operator $\hat{A}$, representing observable A, has to normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two normalized eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5\,, \quad \psi_2 = (4\phi_1 - 3\phi_2)/5\,. \tag{1}$$

  1. Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately) after measurement?
  2. if B is now measured, what are the possible results, and what are their probabilities?
  3. Right after the measurement of B, A is measured again. What is the probability of getting a_1?

Harmonic oscillator

- Discuss how to solve (=find eigenfunctions and eigenvalues of the Hamiltonian operator) the harmonic oscillator with operatorial methods.

- A particle of mass m in the harmonic oscillator potential starts out in the state:

$$\psi(x,0) = A\left(1 - 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2}\,,$$

  for some constant A.

  1. Express $\psi(x,0)$ as a linear combination eigenfunctions of the Hamiltonian.
  2. Find A.

3. What is the expectation value of the energy?
4. At some later time T the wave function is

$$\psi(x,T) = B\left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x\right)^2 e^{-\frac{m\omega}{2\hbar}x^2},$$

for some constant B. What is the smallest possible value of T?

The free particle

- Discuss the Schrödinger equation for a free particle, the problem of normalizing the eigenstates of the Hamiltonian operator and its solution in terms of wave-packets.

- A free particle has the initial wave fuction

$$\Psi(x,0) = Ae^{-ax^2}, \tag{2}$$

where A and a are constants (a is real and positive).

1. Normalize $\Psi(x,0)$.
2. Find $\Psi(x,t)$.
   Hint: Integrals of the form

$$\int_{-\infty}^{+\infty} e^{-(ax^2+bx)}dx \tag{3}$$

   can be handled by "completing the square": Let $y \equiv \sqrt{a}[x + (b/2a)]$, and note that $(ax^2 + bx) = y^2 - (b^2/4a)$.
   Answer:

$$\Psi(x,t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{-ax^2/[1+(2i\hbar at/m)]}}{\sqrt{1+(2i\hbar at/m)}}. \tag{4}$$

3. Find $|\Psi(x,t)|^2$. Express your answer in terms of the quantity

$$w \equiv \sqrt{\frac{a}{1+(2\hbar at/m)^2}}. \tag{5}$$

   Sketch $|\Psi(x,t)|^2$ (as a function of x) at $t = 0$ and again for some large t. Qualitatevely, what happens to $|\Psi|^2$, as time goes on?

4. Find $\langle x \rangle$, $\langle p \rangle$,$\langle x^2 \rangle$,$\langle p^2 \rangle$, σ_x and σ_p.
   Partial answer: $\langle p^2 \rangle = a\hbar^2$, but it might take some algebra to put it in this simple form.
5. Does the uncertainty principle hold? At what time t does the system come closest to the uncertainty limit?
